Solving geometry problems via mechanical principles

MAN Yiu Kwong

Department of Mathematics
Hong Kong Institute of Education
10 Lo Ping Road, Tai Po, NT
HONG KONG, S.A.R., CHINA

Email: ykman@ied.edu.hk

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Contents

- Abstract
- A historical example
- An Area problem of a triangle
- Final remarks
- References

Abstract

The application of physical principles in solving mathematics problems have often been neglected in the teaching of physics or mathematics, especially at the secondary school level. This paper discusses how to apply the mechanical principles to geometry problems via concrete examples, which aims at providing insight and inspirations to physics or mathematics teachers that physics can be a useful tool in mathematics problem solving.

A historical example

A famous example on using mechanical principle to solve geometry problems was provided by Archimedes (287-212 BC). Using the principle of lever, he was able to derive the correct formula for the volume of a sphere (Man & Lo, 2002). Let us illustrate how such an approach works. In Figure 1, N denotes the origin of a plane and TS is a horizontal line through N. Let TN = NS = 2r. First, we draw a circle with radius r, a square ABCD with side 2r and an isosceles triangle NEF with height 2r and base 4r. Then, the whole figure is rotated about TS to generate a sphere, a cylinder and a cone. Next, an arbitrary vertical thin slice of thickness Δx at a distance x from N, is cut from the three solids in turn. By simple calculations, we have:

The volume of slice from the sphere = $\pi(2xr - x^2)\,\Delta x$

The volume of slice from the cylinder = $\pi r^2 \Delta x$

The volume of slice from the cone = $\pi x^2 \Delta x$

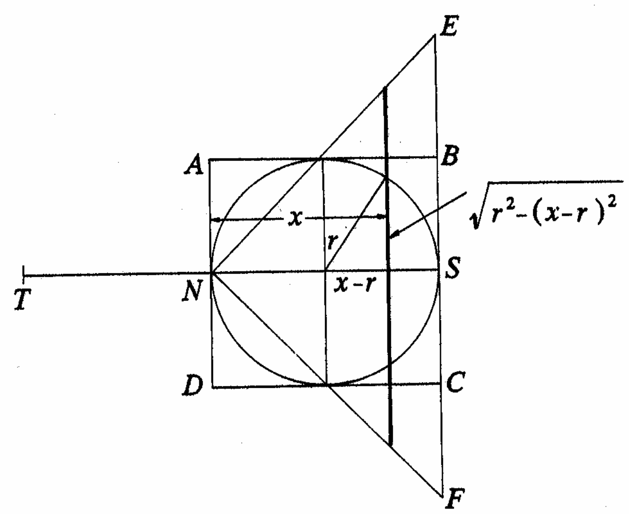

Figure 1

Consider TS as a lever with pivot at N. By hanging the slices from the sphere and the cone at the point T, the total moment generated is:

$$[\pi(2xr - x^2)\Delta x + \pi x^2 \Delta x] \times 2r = 4\pi r^2 x\,\Delta x\ .$$

It is equal to four times the moment generated by the slice of the cylinder on the opposite side of the lever. Since this relation holds for arbitrary slices at any position x on NS , so we have:

(volume of sphere + volume of cone) × 2r = (volume of cylinder) × 4r.

Let V denotes the volume of sphere. Then, we obtain:

$$(V + \frac{8\pi r^3}{3}) \times 2r = 8\pi r^4,$$

and hence $V = \frac{4}{3}\pi r^3$. We can see that it is a correct formula for the volume of sphere and the approach adopted is quite ingenious, without using any sophisticated mathematics.

An area problem of a triangle

The second example is concerned with an area problem of a triangle, as shown in Figure 2. If the area of ΔABC is a cm2 and the points L, M, N divide the line segments BC, CA and AB in the ratio of, respectively, what is the area of ΔPQR?

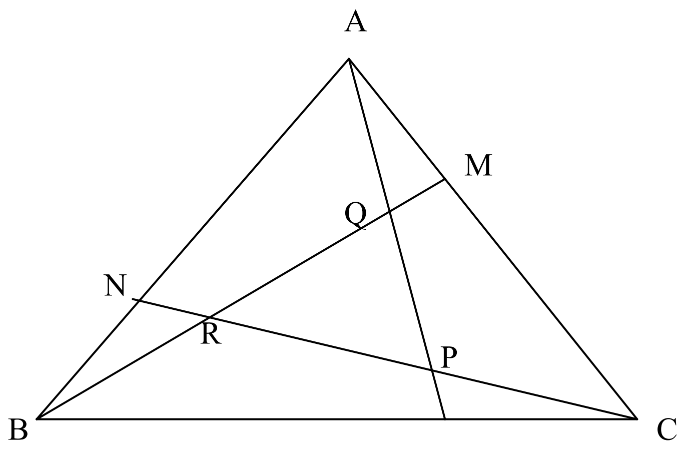

Figure 2

We illustrate how to use the principle of equilibrium to solve this problem. First, let us assume there are three very small solids, with masses 1g, 2g and 4g, located at the points A, B and C respectively. Since AN : NB = 2 : 1, so N is the center of gravity of AB, with a mass of 3g. Similarly, L is the center of gravity of BC, with a mass of 6g.

Now, the center of gravity of must lie on both CN and AL, which means at P. For equilibrium, the moment of the masses at A and L about P must be equal, so 1 × AP = 6 × LP. Hence, AP : LP = 6 : 1. Since BL : LC = 2 : 1, so $S_{\Delta APC} = \frac{6}{7} \times S_{\Delta ACL} = \frac{6}{7} \times \frac{1}{3} \times S_{\Delta ABC} = \frac{2a}{7} cm^2$.

Using the same arguments, we have $S_{\Delta BCR} = S_{\Delta ABQ} = \frac{2a}{7} cm^2$. Therefore,

$$S_{\Delta PQR} = S_{\Delta ABC} - S_{\Delta ABQ} - S_{\Delta BCR} - S_{\Delta CAP} = a - 3 \times \frac{2a}{7} = \frac{a}{7} cm^2$$

If this problem is modified to find the ratio of the area of ΔPQR to that of ΔABC, with AN : NB = NL : LC = CM : MA = m : n [1], we can see that this method works equally well.

Final remarks

There is a common perception that the relationship between mathematics and physics is very close because mathematics is a basic tool in studying physics. However, the use of physical principles or approaches in solving mathematics problems has often been neglected by teachers or students, especially at the secondary level. We hope the examples presented in this paper could bring some insight and inspirations to physics or mathematics teachers that physics can also be a useful tool in mathematics problem solving too. More examples and applications of physics to geometry can be found in the book written by Wen-Jun Wu (吳文俊, 2003).

[1] The answer will be $\frac{(n-m)^3}{n^3 - m^3}$

References

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2. 吳文俊 (2003).《力學在幾何中的一些應用》.香港：智能教育出版社.